Alternative Connectives for Classical Propositional Logic

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June 29, 2016

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Syntactic derivability: $\Gamma \vdash \psi$
  - a proof tree with conclusion ψ and open assumptions Γ

Example: a set of rules (Minimal Logic + Double Negation)

$$\dfrac{\begin{array}{c}[\phi]^u\\ \mathcal{D}\\ \psi\end{array}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \phi \to \psi & \phi\end{array}}{\psi} \to_e \qquad \dfrac{}{\neg\neg\phi \to \phi}\ \mathrm{DN}$$

Where $\neg\phi := \phi \to \bot$.

Syntax: Natural Deduction

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$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \to \psi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \phi\end{matrix}}{\psi} \to_e \qquad \dfrac{}{\neg\neg\phi \to \phi}\ \mathrm{DN}$$

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Where $\neg\phi := \phi \to \bot$.

Syntax: Natural Deduction

Example: Peirce Law

$$\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\bot}{\neg\neg p}\,\to_i, v}{p}\,\to_e}{((p \to q) \to p) \to p}\,\to_i, u$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

Example: Peirce Law

$$\neg p \qquad p \quad \rightarrow_e$$

$$\bot \quad \rightarrow_i, v$$

$$\neg\neg p \rightarrow p \ \text{DN} \qquad \neg\neg p \quad \rightarrow_e$$

$$p \quad \rightarrow_i, u$$

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

$$\{\neg p, p\} \vdash q$$

$$\{\neg p\} \vdash p \rightarrow q$$

$$\{(p \rightarrow q) \rightarrow p\} \vdash \neg\neg p$$

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Example: Peirce Law

$$\frac{\neg p \qquad p}{\bot} \rightarrow_e$$

$$\frac{\bot}{\neg\neg p} \rightarrow_i, v$$

$$\frac{\frac{}{\neg\neg p \rightarrow p}\,\mathrm{DN} \qquad \neg\neg p}{p} \rightarrow_e$$

$$\frac{p}{((p \rightarrow q) \rightarrow p) \rightarrow p} \rightarrow_i, u$$

$\{\neg p, p\} \vdash q$

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Example: Peirce Law

$$\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p \qquad \dfrac{}{p \to q}\to_i}{p}\to_e}{\bot}\to_e}{\neg\neg p}\to_i, v}{\dfrac{p}{((p \to q) \to p) \to p}\to_i, u}\to_e$$

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Example: Peirce Law

$$
\dfrac{\dfrac{}{\neg\neg p \to p}\ \mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{v} \qquad \dfrac{(p \to q) \to p^{u} \qquad \dfrac{q}{p \to q}\ {\to_i}, w}{p}\ {\to_e}}{\bot}\ {\to_e}}{\neg\neg p}\ {\to_i}, v}{\dfrac{p}{((p \to q) \to p) \to p}\ {\to_i}, u}\ {\to_e}
$$

$$
\begin{gathered}
\{\neg p, p\} \vdash q \\
\{\neg p\} \vdash p \to q \\
\{(p \to q) \to p\} \vdash \neg\neg p \\
\emptyset \vdash ((p \to q) \to p) \to p
\end{gathered}
$$

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  \dfrac{}{\neg\neg p \to p}\,\mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{
      \neg p^{\,v}
      \qquad
      \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{q}{p \to q}\,\to_i, w}{p}\,\to_e
    }{\bot}\,\to_e
  }{\neg\neg p}\,\to_i, v
}{
  \dfrac{p}{((p \to q) \to p) \to p}\,\to_i, u
}\,\to_e
$$

$$\{\neg p, p\} \vdash q$$
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Syntax: Natural Deduction

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$$

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Example: Peirce Law

$$\dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{q}{p \to q}\to_i, w}{p}\to_e$$

$$\dfrac{\neg p^{\,v} \qquad p}{\bot}\to_e$$

$$\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\bot}{\neg\neg p}\to_i, v}{\dfrac{p}{((p \to q) \to p) \to p}\to_i, u}\to_e$$

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      \qquad
      \dfrac{
        (p \to q) \to p^{\,u}
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{}{\neg\neg q \to q}\,\mathrm{DN}
            \qquad
            \dfrac{
              \dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\,\to_e
            }{\neg\neg q}\,\to_i
          }{q}\,\to_e
        }{p \to q}\,\to_i, w
      }{p}\,\to_e
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$$\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\ \to_e$$

$$\dfrac{\bot}{\neg\neg q}\ \to_i$$

$$\dfrac{}{\neg\neg q \to q}\ \mathrm{DN}$$

$$\dfrac{\neg\neg q \to q \qquad \neg\neg q}{q}\ \to_e$$

$$\dfrac{q}{p \to q}\ \to_i, w$$

$$\dfrac{(p \to q) \to p^{\,u} \qquad p \to q}{p}\ \to_e$$

$$\dfrac{\neg p^{\,v} \qquad p}{\bot}\ \to_e$$

$$\dfrac{\bot}{\neg\neg p}\ \to_i, v$$

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Syntax: Natural Deduction

Example: Peirce Law

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DN}
\UnaryInfC{$\neg\neg p \to p$}
\AxiomC{$\neg p^{v}$}
\AxiomC{$(p \to q) \to p^{u}$}
\AxiomC{}
\RightLabel{DN}
\UnaryInfC{$\neg\neg q \to q$}
\AxiomC{$\neg p^{v}$}
\AxiomC{p^{w}}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i$}
\UnaryInfC{$\neg\neg q$}
\RightLabel{$\to_e$}
\BinaryInfC{q}
\RightLabel{$\to_i, w$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to_e$}
\BinaryInfC{p}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i, v$}
\UnaryInfC{$\neg\neg p$}
\RightLabel{$\to_e$}
\BinaryInfC{p}
\RightLabel{$\to_i, u$}
\UnaryInfC{$((p \to q) \to p) \to p$}
\end{prooftree}
$$

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    \qquad
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      \dfrac{
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        \qquad
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          (p \to q) \to p^{\,u}
          \qquad
          \dfrac{
            \dfrac{
              \dfrac{}{\neg\neg q \to q}\,\mathrm{DN}
              \qquad
              \dfrac{
                \dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\to_e
              }{\neg\neg q}\to_i
            }{q}\to_e
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- Semantic entailment: $\Gamma \vDash \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if Γ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

Example: Classical Minimal Semantics

$[\![\phi]\!]$	$[\![\psi]\!]$	$[\![\phi \to \psi]\!]$
0	0	1
0	1	1
1	0	0
1	1	1

and $[\![\bot]\!] := 0$

Semantics: Classical

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and $[\![\bot]\!] := 0$

Semantics: Classical

Example: Peirce Law

$[\![p]\!]$	$[\![q]\!]$	$[\![p \to q]\!]$	$[\![(p \to q) \to p]\!]$	$[\![((p \to q) \to p) \to p]\!]$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

- $\{q\} \models p \to q$,
- $\{p\} \models (p \to q) \to p$,
- $\emptyset \models ((p \to q) \to p) \to p$.

Semantics: Classical

Example: Peirce Law

$[\![p]\!]$	$[\![q]\!]$	$[\![p \to q]\!]$	$[\![(p \to q) \to p]\!]$	$[\![((p \to q) \to p) \to p]\!]$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	1
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- $\{q\} \vDash p \to q$,
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Example: Peirce Law

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- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
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Preserving Translations

Let **A** and **B** be two natural deduction systems.

Definition

A function t is a *truth-preserving* translation function if and only if:

- given some formula ϕ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

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A function t is a *provability-preserving* translation function if and only if:

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Preserving Translations

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Let $\mathcal{F}_M$ be the smallest set s.t.:

$$a \in \mathcal{F}_M,$$
$$\bot \in \mathcal{F}_M,$$
$$\phi \to \psi \in \mathcal{F}_M,$$

where $\phi, \psi \in \mathcal{F}_M$.

N.b. minimal formulas directly map to classical formulas.

Example: Standard Formulas

Let $\mathcal{F}_S$ be the smallest set s.t.:

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Example: Standard to Minimal Translation

Let $t : \mathcal{F}_S \to \mathcal{F}_M$ be:

$$\begin{aligned} t(\top) &:= \bot \to \bot, \\ t(\neg\phi) &:= t(\phi) \to \bot, \\ t(\phi \wedge \psi) &:= (t(\psi) \to (t(\psi) \to t(\phi)) \to \bot) \to \bot, \\ t(\phi \vee \psi) &:= (t(\phi) \to t(\psi)) \to t(\psi). \end{aligned}$$

t is a truth-preserving and a provability-preserving translation.

Example: Truth-preserving

$[\![\phi]\!]$	$[\![\psi]\!]$	$[\![p \vee q]\!]$	$[\![t(p \vee q)]\!]$	$[\![p \wedge q]\!]$	$[\![t(p \wedge q)]\!]$	etc.
0	0	0	0	0	0	
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Preserving Translations

Example: Provability-preserving

$$\dfrac{\begin{matrix}\mathcal{D}_l & \mathcal{D}_r \\ \phi & \psi\end{matrix}}{\phi \wedge \psi} \wedge_i \quad \Rightarrow \quad \begin{matrix}\mathcal{D}'_l & \mathcal{D}'_r \\ t(\phi) & t(\psi) \\ \multicolumn{2}{c}{\mathcal{D}'} \\ \multicolumn{2}{c}{t(\phi \wedge \psi)}\end{matrix}$$

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Example: Provability-preserving

$$\dfrac{\begin{matrix}\mathcal{D}_l \\ \phi\end{matrix} \quad \begin{matrix}\mathcal{D}_r \\ \psi\end{matrix}}{\phi \wedge \psi} \wedge_i \qquad \Rightarrow \qquad \begin{matrix} \begin{matrix}\mathcal{D}'_l \\ t(\phi)\end{matrix} \quad \begin{matrix}\mathcal{D}'_r \\ t(\psi)\end{matrix} \\ \mathcal{D}' \\ (t(\psi) \to (t(\psi) \to t(\phi)) \to \bot) \to \bot \end{matrix}$$

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$$\dfrac{\begin{array}{c}\mathcal{D}_l\\ \phi\end{array} \quad \begin{array}{c}\mathcal{D}_r\\ \psi\end{array}}{\phi \wedge \psi}\wedge_i \qquad \Rightarrow \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}'_r\\ \psi\\ \mathcal{D}'_1\\ (\psi \to \phi) \to \bot\end{array} \quad \begin{array}{c}\mathcal{D}'_l\\ \phi\\ \mathcal{D}'_2\\ \psi \to \phi\end{array}}{\bot}\to_e}{(\psi \to (\psi \to \phi) \to \bot) \to \bot}\to_i, u$$

Preserving Translations

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$$
\dfrac{\begin{matrix}\mathcal{D}_l\\ \phi\end{matrix}\qquad\begin{matrix}\mathcal{D}_r\\ \psi\end{matrix}}{\phi\wedge\psi}\wedge_i
\qquad\Rightarrow\qquad
\dfrac{\dfrac{\dfrac{\psi\to(\psi\to\phi)\to\bot^{u}\qquad\begin{matrix}\mathcal{D}'_r\\ \psi\end{matrix}}{(\psi\to\phi)\to\bot}\to_e\qquad\begin{matrix}\mathcal{D}'_l\\ \phi\\ \mathcal{D}'_2\\ \psi\to\phi\end{matrix}}{\bot}\to_e}{(\psi\to(\psi\to\phi)\to\bot)\to\bot}\to_i, u
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$$

Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

Example: duality of formulas

tautology	contradiction
$\wedge$	$\vee$
$\vee$	$\wedge$
$\neg$	$\neg$
$\rightarrow$	?
?	$\rightarrow$

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\quad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
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Arrow Formulas

Example: Arrow Formulas

Let $\mathcal{F}_A$ be the smallest set such that:

$$a \in \mathcal{F}_A,$$

$$\phi \to \psi \in \mathcal{F}_A,$$

$$\phi \not\leftarrow \psi \in \mathcal{F}_A,$$

where $\phi, \psi \in \mathcal{F}_A$.

Example: Classical Arrow Semantics

$[\![\phi]\!]$	$[\![\psi]\!]$	$[\![\phi \to \psi]\!]$	$[\![\phi \not\leftarrow \psi]\!]$
0	0	1	0
0	1	1	1
1	0	0	0
1	1	1	0

Example: a set of rules (Arrow Natural Deduction)

Given some $\chi \in \mathcal{F}_A$.

$$\frac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi \to \psi & \phi\end{matrix}}{\psi} \to_{e_1} \qquad \frac{\begin{matrix}\mathcal{D} \\ \neg(\phi \to \psi)\end{matrix}}{\phi} \to_{e_2}$$

$$\frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \neg\phi & \psi\end{matrix}}{\phi \not\leftarrow \psi} \not\leftarrow_i \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\neg\phi} \not\leftarrow_{e_1} \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\psi} \not\leftarrow_{e_2}$$

Where $\neg\phi := \phi \to (\chi \not\leftarrow \chi)$.

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Example: a deduction

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1}
\qquad
\dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}
}{\chi \not\leftarrow \chi}\rightarrow_{e_1}
}{\neg\psi}\rightarrow_{i}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1}
\qquad
\dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}
}{\chi \not\leftarrow \chi}\rightarrow_{e_1}
}{\neg(\psi \rightarrow \psi)}\rightarrow_{i}
}{\psi}\rightarrow_{e_2}
}{
\dfrac{\psi \not\leftarrow \psi}{(\phi \not\leftarrow \phi) \rightarrow (\psi \not\leftarrow \psi)}\rightarrow_{i},u
}\not\leftarrow_{i}
$$

- (Extensionality of $\phi \rightarrow \phi = \psi \rightarrow \psi$, and of $\phi \not\leftarrow \phi = \psi \not\leftarrow \psi$?)

Example: a deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1} \quad \dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}}{\chi \not\leftarrow \chi}\to_{e_1}}{\neg\psi}\to_i \qquad \dfrac{\dfrac{\dfrac{\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1} \quad \dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}}{\chi \not\leftarrow \chi}\to_{e_1}}{\neg(\psi \to \psi)}\to_i}{\psi}\to_{e_2}}{\dfrac{\psi \not\leftarrow \psi}{(\phi \not\leftarrow \phi) \to (\psi \not\leftarrow \psi)}\to_{i}, u}\not\leftarrow_i$$

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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\phi \nleftarrow \phi^u}{\neg\phi}\,{\nleftarrow}_{e_1}
      \qquad
      \dfrac{\phi \nleftarrow \phi^u}{\phi}\,{\nleftarrow}_{e_2}
    }{\chi \nleftarrow \chi}\,{\to}_{e_1}
  }{\neg\psi}\,{\to}_{i}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\phi \nleftarrow \phi^u}{\neg\phi}\,{\nleftarrow}_{e_1}
        \qquad
        \dfrac{\phi \nleftarrow \phi^u}{\phi}\,{\nleftarrow}_{e_2}
      }{\chi \nleftarrow \chi}\,{\to}_{e_1}
    }{\color{red}{\neg(\psi \to \psi)}}\,{\to}_{i}
  }{\color{red}{\psi}}\,{\to}_{e_2}
}{
  \dfrac{\psi \nleftarrow \psi}{(\phi \nleftarrow \phi) \to (\psi \nleftarrow \psi)}\,{\to}_{i},u
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    }{\chi \nleftarrow \chi}\,{\rightarrow_{e_1}}
  }{\color{red}{\neg\psi}}\,{\rightarrow_i}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\phi \nleftarrow \phi^u}{\neg\phi}\,{\nleftarrow_{e_1}}
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  }{\color{red}{\psi}}\,{\rightarrow_{e_2}}
}{
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}\,{\nleftarrow_i}
$$

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Arrow Formulas

Example: a deduction

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\phi \not\leftarrow \phi^{u}}{\neg\phi}\ {\not\leftarrow}_{e_1}
      \qquad
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  \qquad
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- We can show soundness and completeness by truth-preserving and provability-preserving translations.
- We have seen alternative connectives for classical propositional logic.

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